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Linewidth enhancement factor in a microcavity Brillouin laser: supplementary material

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This document provides supplementary information to "Linewidth enhancement factor in a microcavity Brillouin laser." In section 1, we discuss the equivalence of the α factor and the normalized detuning of the Brillouin laser, as well as its connections to the α factor commonly used in semiconductor lasers. In section 2, we review the frequency discriminator measurements used in the main text to determine laser frequency noise. In section 3, we present a full analysis of the Brillouin laser noise, including the effects of pump phase noise. In section 4, a measurement of the α factor enhanced noise measured on a second device is presented. In section 5 measurement of the relative intensity noise of the Brillouin laser is presented. In section 6, we study the effect of the anti-Stokes process on the α-factor linewidth enhancement in a cascaded Brillouin laser system.

1. DERIVATION OF THE ALPHA FACTOR IN BRILLOUIN LASERS

We derive the α factor in stimulated Brillouin laser (SBL) systems by starting from the Hamiltonian of the system:

$$H = \hbar(\omega_P \tilde{A}^\dagger \tilde{A} + \omega_s \tilde{a}^\dagger \tilde{a} + \Omega \tilde{b}^\dagger \tilde{b}) + \hbar g_B (\tilde{A}^\dagger \tilde{a} \tilde{b} + \tilde{A} \tilde{a}^\dagger \tilde{b}^\dagger) \quad \text{(S1)}$$

where $\tilde{A}$, $\tilde{a}$ and $\tilde{b}$ are the lowering operators of the pump, Stokes and phonon modes, respectively; ω_P, ω_s and Ω are the resonance frequencies of the pump, Stokes and phonon modes, respectively; and g_B is the single-particle Brillouin coupling [1]. We have ignored terms that are strongly out of phase match (i.e., energy non-conserving) in the Hamiltonian to simplify the discussion. The fast time dependencies are removed from the operators as follows:

$$A \equiv \tilde{A} \exp(i\omega_{P,in} t) \quad \text{(S2)}$$
$$a \equiv \tilde{a} \exp(i\omega_L t) \quad \text{(S3)}$$
$$b \equiv \tilde{b} \exp(i\Omega_L t) \quad \text{(S4)}$$

where A, a and b are the slow-varying lowering operators; $\omega_{P,in}$ is the pumping frequency; ω_L is the SBL frequency and Ω_L is the mechanical vibration frequency. Replacing the operators with the slow-varying ones results in an effective Hamiltonian:

$$H = \hbar(\delta\omega_P A^\dagger A + \delta\omega a^\dagger a + \delta\Omega b^\dagger b) + \hbar g_B (A^\dagger a b + A a^\dagger b^\dagger) \quad \text{(S5)}$$

where $\delta\omega_P \equiv \omega_P - \omega_{P,in}$ is the pump mode frequency detuning compared to the external pump, and $\delta\omega \equiv \omega_s - \omega_L$ ($\delta\Omega \equiv \Omega - \Omega_L$) is the detuning of Stokes (phonon) cavity mode compared to the laser (mechanical vibration) frequency. We note that the slow-varying amplitudes are directly referenced to the true oscillating frequencies of each mode instead of the resonance frequencies, which removes the fast time dependence in the interaction terms.

The Heisenberg equations of motion for the Stokes mode and the phonon mode are derived. Then, the quantum operators are replaced with classical fields as the dominant source of noise in this system is phonon thermal noise [1]. Finally, phenomenological damping terms are inserted as follows,

$$\frac{da}{dt} = -\left(\frac{\gamma}{2} + i\delta\omega\right) a - i g_B A b^* \quad \text{(S6)}$$
$$\frac{db}{dt} = -\left(\frac{\Gamma}{2} + i\delta\Omega\right) b - i g_B A a^* \quad \text{(S7)}$$

where γ (Γ) is the energy decay rates for the Stokes (phonon) mode.

We first seek nonzero steady-state solutions to the above equations that represent SBLs. By writing the equation for b^* using Eq. (S7),

$$\frac{db^*}{dt} = -\left(\frac{\Gamma}{2} - i\delta\Omega\right) b^* + i g_{\mathrm{B}} A^* a \tag{S8}$$

the equations (S6) and (S8) form a linear system in a and b^*. The requirement for nonzero solutions (i.e., zero determinant of the coefficient matrix) gives the equation:

$$\left(\frac{\gamma}{2} + i\delta\omega\right)\left(\frac{\Gamma}{2} - i\delta\Omega\right) - g_{\mathrm{B}}^2 |A|_0^2 = 0 \tag{S9}$$

where the subscript 0 indicates steady state. This complex equation can be solved as

$$\frac{2\delta\omega}{\gamma} = \frac{2\delta\Omega}{\Gamma} \tag{S10}$$

$$g_{\mathrm{B}}^2 |A|_0^2 = \frac{\gamma\Gamma}{4}\left(1 + \frac{4\delta\Omega^2}{\Gamma^2}\right) \tag{S11}$$

For convenience, we define $\alpha \equiv 2\delta\omega/\gamma = 2\delta\Omega/\Gamma$ and later demonstrate that α is indeed the linewidth enhancement factor. With α defined, the steady-state pump photon number is,

$$|A|_0^2 = \frac{\gamma\Gamma}{4}\frac{1+\alpha^2}{g_{\mathrm{B}}^2} = \frac{\gamma}{2g}(1+\alpha^2) \tag{S12}$$

where the Brillouin gain coefficient $g = 2g_{\mathrm{B}}^2/\Gamma$ has been defined. Since $\Gamma \gg \gamma$ in our microcavity system, we can adiabatically eliminate b^* from Eq. (S6) by setting $db^*/dt = 0$ in Eq. (S8).

$$\frac{da}{dt} = \left(-\frac{\gamma}{2} + \frac{g|A|^2}{1+\alpha^2}\right)(1 + i\alpha)\, a \tag{S13}$$

where the definition of α has been used. Here, $|A|^2$ implicitly depends on a through the pump mode dynamics and controls the gain saturation. Alternatively, Eqn. S13 can be represented using the amplitude $|a|$ and phase $\phi_a = \ln(a/a^*)/(2i)$ variables,

$$\frac{d|a|}{dt} = \left(-\frac{\gamma}{2} + \frac{g|A|^2}{1+\alpha^2}\right)|a| \tag{S14}$$

$$\frac{d\phi_a}{dt} = \left(-\frac{\gamma}{2} + \frac{g|A|^2}{1+\alpha^2}\right)\alpha \tag{S15}$$

which illustrates that $\alpha = |a|\dot{\phi}_a/|\dot{a}|$ represents amplitude-phase coupling.

Henry [2] defined the α factor as the ratio of the change in real part of the refractive index and the change in the imaginary part. Below we show that this interpretation is consistent with that derived from the coupled-mode equations. For a system with Lorentzian gain, the imaginary part of the gain-induced susceptibility can be written as

$$\chi_{\mathrm{I}}(\omega_{\mathrm{B}}) = -\frac{\chi_{\mathrm{B}}}{1 + 4\omega_{\mathrm{B}}^2/\Gamma^2} \tag{S16}$$

where Γ is the gain bandwidth, χ_{B} is a positive constant describing the strength of the gain at the line center, and the angular frequency ω_{B} is referenced to the gain center (i.e., detuning relative to gain center). By the Kramers-Kronig relations, χ_{I} necessarily leads to the real part of the susceptibility χ_{R} through the relation,

$$\chi_{\mathrm{R}}(\omega_{\mathrm{B}}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\chi_{\mathrm{I}}(\omega'_{\mathrm{B}})}{\omega'_{\mathrm{B}} - \omega_{\mathrm{B}}} d\omega'_{\mathrm{B}} = \chi_{\mathrm{B}}\frac{2\omega_{\mathrm{B}}/\Gamma}{1 + 4\omega_{\mathrm{B}}^2/\Gamma^2} \tag{S17}$$

The refractive index can be written as $n(\omega_{\mathrm{B}})^2 = n^2 + \chi_{\mathrm{R}} + i\chi_{\mathrm{I}}$, where n is the material refractive index (dispersion in n has been ignored). Assuming $\chi_{\mathrm{B}} \ll n^2$, we can find the real part n' and imaginary part n'' of the refractive index:

$$n' = n + \frac{\chi_{\mathrm{R}}}{2n} = n + \frac{\chi_{\mathrm{B}}}{2n}\frac{2\omega_{\mathrm{B}}/\Gamma}{1 + 4\omega_{\mathrm{B}}^2/\Gamma^2} \tag{S18}$$

$$n'' = \frac{\chi_{\mathrm{I}}}{2n} = -\frac{\chi_{\mathrm{B}}}{2n}\frac{1}{1 + 4\omega_{\mathrm{B}}^2/\Gamma^2} \tag{S19}$$

The α factor can then be obtained as,

$$\alpha = -\frac{\partial n'/\partial\chi_{\mathrm{B}}}{\partial n''/\partial\chi_{\mathrm{B}}} = \frac{2\omega_{\mathrm{B}}}{\Gamma} \tag{S20}$$

Setting $\omega_{\mathrm{B}} = \delta\Omega$ recovers the desired result, $\alpha = 2\delta\Omega/\Gamma$. There are different conventions regarding the sign of α, and here we choose the negative sign which would be consistent with the $\exp(-i\omega t)$ phasor used throughout.

To further establish the connection of α to linewidth broadening, the SBL linewidth is derived. We will again assume $\Gamma \gg \gamma$ and defer the more general case to Section 3. For this analysis we add classical noise terms to Eqs. (S6) and (S7),

$$\frac{da}{dt} = -\frac{\gamma}{2}(1 + i\alpha)\, a - i g_{\mathrm{B}} A b^* + f_a(t) \tag{S21}$$

$$\frac{db}{dt} = -\frac{\Gamma}{2}(1 + i\alpha)\, b - i g_{\mathrm{B}} A a^* + f_b(t) \tag{S22}$$

where f_a and f_b are classical noise operators for the Stokes and phonon mode, respectively, satisfying the following correlations:

$$\langle f_a^*(t+\tau) f_a(t)\rangle = 0 \tag{S23}$$

$$\langle f_b^*(t+\tau) f_b(t)\rangle = n_{\mathrm{th}}\Gamma\delta(\tau) \tag{S24}$$

and n_{th} is the number of thermal quanta in the phonon mode (thermal quanta in the optical modes are negligible at room temperature).

Adiabatically eliminating b gives

$$\frac{da}{dt} = \left(-\frac{\gamma}{2} + \frac{g|A|^2}{1+\alpha^2}\right)(1 + i\alpha)\, a + \tilde{f}_a(t) \tag{S25}$$

$$\tilde{f}_a \equiv f_a - \frac{i g_{\mathrm{B}} A}{1 - i\alpha}\frac{2}{\Gamma} f_b^* \tag{S26}$$

where we defined a composite fluctuation term $\tilde{f}_a$ for the SBL. Its correlation reads

$$\begin{aligned}\langle \tilde{f}_a^*(t)\tilde{f}_a(0)\rangle &= \langle f_a^*(t) f_a(0)\rangle + \frac{g_{\mathrm{B}}^2|A|_0^2}{1+\alpha^2}\frac{4}{\Gamma^2}\langle f_b^*(t) f_b(0)\rangle \\ &= n_{\mathrm{th}}\gamma\delta(t)\end{aligned} \tag{S27}$$

which is independent of α. Applying a standard linewidth analysis, the SBL linewidth is found as,

$$\Delta\omega_{\mathrm{SBL}} = \frac{\gamma}{2N_a} n_{\mathrm{th}}\left(1 + \alpha^2\right) \tag{S28}$$

where $N_a = |a|^2$ is the steady-state photon number in the Stokes mode. This is readily shown to agree with Eq. (3) in the main text in the limit of $\Gamma \to \infty$ when expressed in terms of output SBL power.

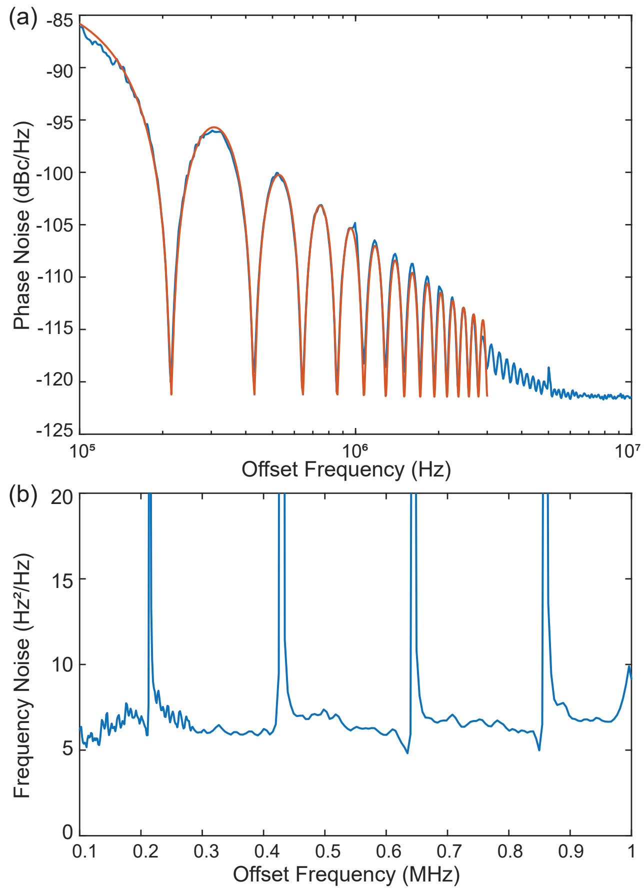

Fig. S1. SBL noise measurement and fitting. (a) Blue curve is the measured phase noise spectrum from the self-heterodyne output when pump wavelength is 1538 nm and SBL power is 1.29 mW. Red curve is the fitting according to Eq. (S33) to obtain the frequency noise S. (b) The converted frequency noise spectrum from panel (a).

2. FREQUENCY DISCRIMINATOR MEASUREMENTS

In this section, the frequency noise measurement is studied to arrive at the transfer function that relates the measured phase noise spectrum (see Fig. S1 (a)) to the frequency noise spectral density S plotted in Fig. 3 of the Main Text. As shown in the Supplement Information of our previous work [3], pump noise conversion is believed to be the dominant noise source at low offset frequency, while white Schawlow-Townes-like noise dominants at high offset frequency (usually over 100 kHz).

For white frequency noise, the correlation of the time derivative of the phase satisfies,

$$\langle \dot{\phi}(t+\tau)\dot{\phi}(t)\rangle = \Delta\omega_{\mathrm{N}}\delta(\tau) \tag{S29}$$

where $\Delta\omega_{\mathrm{N}}$ is the Lorentzian full-width-at-half-maximum linewidth in rad/s, including both fundamental ($\Delta\omega_{\mathrm{SBL}}$) and technical contributions. The two-sided spectral density function for the instantaneous frequency $\nu \equiv \dot{\phi}/(2\pi)$ is given by the Fourier transform of the correlation function:

$$S_{\mathrm{w}} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\langle \dot{\phi}(t+\tau)\dot{\phi}(t)\rangle e^{-2\pi i f\tau}d\tau = \frac{\Delta\omega_{\mathrm{N}}}{4\pi^2} \tag{S30}$$

where S_{w} is the white frequency noise spectral density as in the main text.

On account of the time delayed path in the frequency discrimination system, the detected output returns a signal with a noisy phase $\phi(t+\tau)-\phi(t)$, where τ is the interferometer delay. We are thus interested in the frequency noise of $\nu(\tau) \equiv (\dot{\phi}(t+\tau)-\dot{\phi}(t))/(2\pi)$. By the time-shifting property of the Fourier transform,

$$S_{\nu(\tau)}(f) = S_{\mathrm{w}}(2-e^{2\pi i f\tau}-e^{-2\pi i f\tau}) = 4\sin^2(\pi f\tau)S_{\mathrm{w}} \tag{S31}$$

The detected output from the self-heterodyne interferometer is analyzed by a phase noise analyzer. Therefore, converting to phase noise gives,

$$S_{\phi(\tau)}(f) = \frac{1}{f^2}S_{\nu(\tau)}(f) = 4\frac{\sin^2(\pi f\tau)}{f^2}S_{\mathrm{w}} \tag{S32}$$

A typical measured phase-noise spectrum is shown in Fig. S1. In fitting the spectrum, there is both the sinc^2-shaped noise spectrum contributed by the SBL laser, and a noise floor contributed by the photodetector noise equivalent power (NEP). Thus, the following equation is used to describe the total phase noise,

$$S_{\mathrm{Total},\phi}(f) = S_{\mathrm{NEP}} + 4\pi^2\tau^2\mathrm{sinc}^2(\pi f\tau)S_{\mathrm{w}} \tag{S33}$$

where $\mathrm{sinc}(z) \equiv \sin z/z$, $S_{\mathrm{Total},\phi}(f)$ is the total measured phase noise, and S_{NEP} is the NEP contributed phase noise (determined by averaging the measured phase noise between 8 MHz to 10 MHz). S_{w} and the time delay τ are fitting parameters in the measurement (the fiber delay has around 1 km length and therefore provides an approximate delay of $\tau \approx 4.67$ μs). The fitting is performed within the frequency range between 0.1 MHz and 3 MHz, since technical noise becomes significant below 0.1 MHz, while the fringe contrast is reduced for frequencies higher than 3 MHz on account of reduced resolution.

To explicitly illustrate the measured noise is approximately white over this frequency range, we convert the phase noise from discriminator measurement to frequency noise by dividing out the response function, $4\pi^2\tau^2\mathrm{sinc}^2(\pi f\tau)$. As shown in Fig. S1 (b), the overall frequency noise is nearly white except for some spikes resulting from zeros in the response function in combination with the NEP noise contributions.

3. FULL ANALYSIS OF THE BRILLOUIN LASER NOISE

In this section, a more complete analysis of the SBL frequency noise is presented that includes both the effect of the pumping noise and also does not make the adiabatic approximation (i.e., $\Gamma \gg \gamma$). The equations of motion for the Stokes, phonon and pump mode amplitudes, with damping and pumping terms, are:

$$\frac{da}{dt} = -\frac{\gamma}{2}(1+i\alpha)\,a - ig_{\mathrm{B}}Ab^* \tag{S34}$$

$$\frac{db}{dt} = -\frac{\Gamma}{2}(1+i\alpha)\,b - ig_{\mathrm{B}}Aa^* \tag{S35}$$

$$\frac{dA}{dt} = -\left(\frac{\gamma}{2}+i\delta\omega_{\mathrm{P}}\right)A - ig_{\mathrm{B}}ab + \sqrt{\kappa}A_{\mathrm{in}} \tag{S36}$$

where the pump and Stokes mode have the same decay rate γ, κ is the external coupling rate, $A_{\mathrm{in}} > 0$ is the external pumping amplitude (normalized to photon rate), and the other symbols have the same meaning as in Section 1.

It is convenient to work with amplitude ($|a|$, $|b|$, $|A|$) and phase ($\phi_a = \ln(a/a^*)/(2i)$, similar definitions for ϕ_b and ϕ_A)

variables. Their equations can be rewritten as,

$$\frac{d|a|}{|a|dt} = -\frac{\gamma}{2} + g_{\mathrm{B}}\frac{|A||b|}{|a|}\sin\theta \tag{S37}$$

$$\frac{d|b|}{|b|dt} = -\frac{\Gamma}{2} + g_{\mathrm{B}}\frac{|A||a|}{|b|}\sin\theta \tag{S38}$$

$$\frac{d|A|}{|A|dt} = -\frac{\gamma}{2} - g_{\mathrm{B}}\frac{|a||b|}{|A|}\sin\theta + \sqrt{\kappa}\frac{A_{\mathrm{in}}}{|A|}\cos\phi_A \tag{S39}$$

$$\frac{d\phi_a}{dt} = -\frac{\gamma}{2}\alpha - g_{\mathrm{B}}\frac{|A||b|}{|a|}\cos\theta \tag{S40}$$

$$\frac{d\phi_b}{dt} = -\frac{\Gamma}{2}\alpha - g_{\mathrm{B}}\frac{|A||a|}{|b|}\cos\theta \tag{S41}$$

$$\frac{d\phi_A}{dt} = -\delta\omega_{\mathrm{P}} - g_{\mathrm{B}}\frac{|a||b|}{|A|}\cos\theta - \sqrt{\kappa}\frac{A_{\mathrm{in}}}{|A|}\sin\phi_A \tag{S42}$$

where we defined the phase difference $\theta = \phi_A - \phi_a - \phi_b$. The steady-state solutions (indicated by a subscript 0) are given by,

$$\cos\theta_0 = -\frac{\alpha}{\sqrt{1+\alpha^2}} \tag{S43}$$

$$\sin\theta_0 = \frac{1}{\sqrt{1+\alpha^2}} \tag{S44}$$

$$|A|_0^2 = \frac{\gamma}{2g}\left(1+\alpha^2\right) \tag{S45}$$

$$|b|_0^2 = \frac{\gamma}{\Gamma}N_a \tag{S46}$$

$$\sqrt{\kappa}A_{\mathrm{in}}\cos\phi_{A,0} = |A|_0\left(\frac{\gamma}{2} + \frac{gN_a}{1+\alpha^2}\right) \tag{S47}$$

$$\delta\omega_{\mathrm{P}} = \frac{\alpha}{1+\alpha^2}gN_a - \sqrt{\kappa}\frac{A_{\mathrm{in}}}{|A|_0}\sin\phi_{A,0} \tag{S48}$$

where we used the definition $g = 2g_{\mathrm{B}}^2/\Gamma$. Also, although we expressed everything in terms of SBL photon numbers $N_a \equiv |a|_0^2$, it is the input amplitude A_{in} that determines N_a.

Because the pump mode is Pound-Drever-Hall (PDH) locked to the cavity resonance $\phi_{A,0} = 0$. Thus, the input amplitude and detuning can be further simplified as

$$\sqrt{\kappa}A_{\mathrm{in},0} = \left(\frac{\gamma}{2} + \frac{gN_a}{1+\alpha^2}\right)\sqrt{\frac{\gamma}{2g}}\sqrt{1+\alpha^2} \tag{S49}$$

$$\delta\omega_{\mathrm{P},0} = \frac{\alpha}{1+\alpha^2}gN_a \tag{S50}$$

We note that the $\delta\omega_{\mathrm{P},0}$ obtained here is, up to zeroth order of γ/Γ, equal to the negative of beatnote change between the pump and SBL signals induced by amplitude-phase coupling, as measured in Fig. 2a in the main text.

After the steady-state solutions are obtained, the dynamical equations are linearized by defining relative amplitude change variables (e.g., $\delta a = |a|/|a|_0 - 1$) and phase change variables (e.g., $\delta\phi_a = \phi_a - \phi_{a,0}$). Also, Langevin terms are added to the right side of the equations. These are, as before, classical and include only the thermal noise contributions. The linearized equations with noise terms are:

$$\frac{d\delta a}{dt} = \frac{\gamma}{2}(\delta A + \delta b - \delta a - \alpha\delta\theta) + f_{\delta a} \tag{S51}$$

$$\frac{d\delta b}{dt} = \frac{\Gamma}{2}(\delta A + \delta a - \delta b - \alpha\delta\theta) + f_{\delta b} \tag{S52}$$

$$\frac{d\delta A}{dt} = -\frac{\gamma}{2}\delta A - \frac{gN_a}{1+\alpha^2}(\delta a + \delta b - \alpha\delta\theta) \tag{S53}$$

$$\frac{d\delta\phi_a}{dt} = \frac{\gamma}{2}(\alpha\delta A + \alpha\delta b - \alpha\delta a + \delta\theta) + f_{\delta\phi,a} \tag{S54}$$

$$\frac{d\delta\phi_b}{dt} = \frac{\Gamma}{2}(\alpha\delta A + \alpha\delta a - \alpha\delta b + \delta\theta) + f_{\delta\phi,b} \tag{S55}$$

$$\begin{aligned}\frac{d\delta\phi_A}{dt} &= \frac{gN_a}{1+\alpha^2}(\alpha\delta a + \alpha\delta b - \alpha\delta A + \delta\theta) \\ &\quad - \left(\frac{\gamma}{2} + \frac{gN_a}{1+\alpha^2}\right)(\delta\phi_A + f_{\delta\phi,A})\end{aligned} \tag{S56}$$

where f_z represents noise input to the variable z. It is convenient to switch to the frequency domain using $d/dt \to i\omega$. The power spectral density of each noise term can be written as,

$$S_{f,\delta a} = S_{f,\delta\phi,a} = 0 \tag{S57}$$

$$S_{f,\delta b} = S_{f,\delta\phi,b} = \frac{n_{\mathrm{th}}}{2}\frac{\Gamma}{|b|_0^2} \tag{S58}$$

$$S_{f,\delta\phi,A} = S_{\phi,\mathrm{Pump}} \tag{S59}$$

where $S_{\phi,\mathrm{Pump}}$ is the input phase noise contributed by the pump, and each noise term is independent of others. We have ignored the relative intensity noise of the pump, but it can also be analyzed similarly.

The above linear equations can be directly inverted, and the solution for $\delta\phi_a$ is, to the lowest order in ω,

$$\begin{aligned}\delta\phi_a &= \frac{i}{(\gamma+\Gamma)\omega}\left(\alpha\Gamma f_{\delta a} - \alpha\gamma f_{\delta b} - \Gamma f_{\delta\phi,a} + \gamma f_{\delta\phi,b}\right) \\ &\quad - \frac{\gamma}{\gamma+\Gamma}f_{\delta\phi,A}\end{aligned} \tag{S60}$$

where the lowest order of ω approximation remains valid when $\omega \ll \gamma$. From here we obtain the phase noise of the SBL,

$$\begin{aligned}S_{\phi,\mathrm{SBL}} &= \frac{\alpha^2\Gamma^2 S_{f,\delta a} + \alpha^2\gamma^2 S_{f,\delta b} - \Gamma^2 S_{f,\delta\phi,a} + \gamma^2 S_{f,\delta\phi,b}}{(\gamma+\Gamma)^2\omega^2} \\ &\quad + \left(\frac{\gamma}{\gamma+\Gamma}\right)^2 S_{f,\delta\phi,A}\end{aligned} \tag{S61}$$

$$S_{\phi,\mathrm{SBL}} = \frac{\Gamma^2\left(1+\alpha^2\right)}{(\gamma+\Gamma)^2\omega^2}\frac{\gamma}{2N_a}n_{\mathrm{th}} + \left(\frac{\gamma}{\gamma+\Gamma}\right)^2 S_{\phi,\mathrm{Pump}} \tag{S62}$$

Converting to frequency noise gives,

$$S_{\nu,\mathrm{SBL}} = \frac{\Gamma^2\left(1+\alpha^2\right)}{4\pi^2(\gamma+\Gamma)^2}\frac{\gamma}{2N_a}n_{\mathrm{th}} + \left(\frac{\gamma}{\gamma+\Gamma}\right)^2 S_{\nu,\mathrm{Pump}} \tag{S63}$$

Thus, the fundamental linewidth of the SBL is given by

$$\Delta\omega_{\mathrm{SBL}} = \left(\frac{\Gamma}{\gamma+\Gamma}\right)^2\left(1+\alpha^2\right)\frac{\gamma}{2N_a}n_{\mathrm{th}} \tag{S64}$$

Note that the above derivation automatically incorporates non-adiabaticity and the linewidth enhancement factor. Also, the transduction of the pump phase noise is, when the pump mode is PDH locked,

$$S_{\nu,\mathrm{SBL}} = \left(\frac{\gamma}{\gamma+\Gamma}\right)^2 S_{\nu,\mathrm{P}} \tag{S65}$$

and is independent of the α factor.

We briefly comment on the noise behavior when $\delta\omega_{\mathrm{P}}$ is tuned away from its PDH-locked value, which happens because the PDH locking can reduce, but not totally eliminate, the drifting in $\delta\omega_{\mathrm{P}}$. Repeating the previous analyses, we arrive at the following

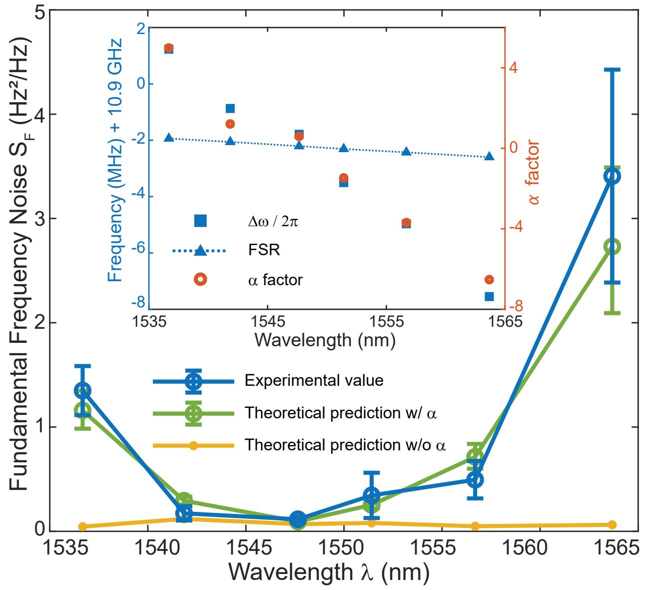

Fig. S2. SBL frequency noise enhancement measured using a second device. Measured SBL fundamental frequency noise S_F (blue); theoretical S_F prediction (green) with α obtained from the plot provided in the inset. The non-enhanced S_0 formula (α=0) prediction (yellow) is also shown. All data are plotted versus pump wavelength and normalized to 1 mW output power. Inset: The extrapolated beating frequency (squares) and FSR (triangles) are plotted versus wavelength. The calculated α factor (red circles) is plotted versus wavelength.

expression for $S_{\nu,\mathrm{SBL}}$ and expand it as a series in the relative variation of detuning:

$$\frac{S_{\nu,\mathrm{SBL}}}{S_{\nu,\mathrm{P}}} \approx \left(\frac{\gamma}{\gamma+\Gamma}\right)^2 + \alpha^2\left(\frac{\gamma}{\gamma+\Gamma}\right)\left(\frac{\delta\omega_\mathrm{P}}{\delta\omega_{\mathrm{P},0}}-1\right) + \frac{\alpha^4}{4}\left(\frac{\delta\omega_\mathrm{P}}{\delta\omega_{\mathrm{P},0}}-1\right)^2 \quad \text{(S66)}$$

where we have kept only the lowest-order term in γ/Γ for each coefficient. As the last term no longer contains the γ/Γ reduction, the phase noise transduction is strongly dependent upon α. Thus an imperfect PDH locking increases the transferred pump phase noise in proportion to α^2.

4. NOISE ENHANCEMENT MEASURED IN A SECOND DEVICE

To verify the generality of our findings, we have performed the experiment on another device and summarized the main data in Fig. S2. Data definition and calculation methods are the same as the main text. In the measurement, we have chosen six longitudinal modes in one transverse mode family. This device has a lowest frequency noise S_F of 0.10 Hz^2/Hz and the measured α factor is as large as 6. The measured noise enhancement is overall in good agreement with the α factor predictions.

5. RELATIVE INTENSITY NOISE

Relative intensity noise (RIN) of the SBL is measured by coupling the laser to a photodetector, filtering out the DC part and analyzing the high-frequency part with an electric spectrum analyzer. A typical RIN measurement is shown in Fig. S3.

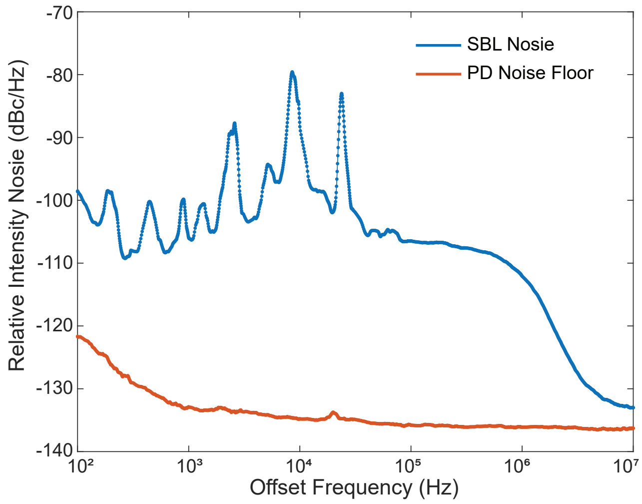

Fig. S3. Typical measured relative intensity noise of SBLs. The output SBL power is 0.31mW at 1550 nm. The intrinsic (coupling) Q factor is 162.7×10^6 (434.5×10^6).

6. ALPHA FACTOR IN CASCADED BRILLOUIN LASERS

Here we study the effect of the anti-Stokes process on the α factor in a cascaded Brillouin laser system. For this analysis, we introduce another pair of modes, a_c and b_c, which are the amplitude of the second-order Stokes mode and the associated phonon mode (normalized to particle numbers), respectively. The cascading Hamiltomian becomes,

$$\begin{aligned} H = {} & \hbar(\delta\omega_\mathrm{P}A^\dagger A + \delta\omega a^\dagger a + \delta\Omega b^\dagger b + \delta\omega_c a_c^\dagger a_c + \delta\Omega_c b_c^\dagger b_c) \\ & + \hbar g_\mathrm{B}(A^\dagger ab + Aa^\dagger b^\dagger) + \hbar g_\mathrm{B}(a^\dagger a_c b_c + a a_c^\dagger b_c^\dagger) \end{aligned} \quad \text{(S67)}$$

where $\delta\omega_c$ ($\delta\Omega_c$) is the detuning of the cascading Stokes (phonon) mode compared to the laser (mechanical vibration) frequency. Although we focus on a single order of cascade, the method can be readily generalized to high-order cascading SBLs.

The classical equations of motion for the cascading Stokes mode and the phonon mode read,

$$\frac{da_c}{dt} = -\left(\frac{\gamma}{2}+i\delta\omega_c\right)a_c - ig_\mathrm{B}ab_c^* \quad \text{(S68)}$$

$$\frac{db_c}{dt} = -\left(\frac{\Gamma}{2}+i\delta\Omega_c\right)b_c - ig_\mathrm{B}aa_c^* \quad \text{(S69)}$$

By the same argument from Section 1, we define an alpha factor for the two detunings:

$$\alpha_c \equiv \frac{2\delta\omega_c}{\gamma} = \frac{2\delta\Omega_c}{\Gamma} \quad \text{(S70)}$$

By obtaining the steady-state solutions and upon substitution into the equations for the first Stokes mode, we find,

$$\frac{da}{dt} = -\left(\frac{\gamma}{2}+i\delta\omega\right)a - ig_\mathrm{B}Ab^* - \frac{gN_{a,c}}{1+\alpha_c^2}(1-i\alpha_c)a \quad \text{(S71)}$$

$$\frac{db}{dt} = -\left(\frac{\Gamma}{2}+i\delta\Omega\right)b - ig_\mathrm{B}Aa^* \quad \text{(S72)}$$

where $N_{a,c} \equiv |a_c|^2$. This extra term introduces extra loss and frequency shift induced by the cascading Stokes wave. This can be made more explicit by rearranging the loss and detuning terms as follows,

$$\frac{da}{dt} = -\left(\frac{\gamma}{2} + \frac{gN_{a,c}}{1+\alpha_c^2}\right) a - i\left(\delta\omega - \frac{gN_{a,c}}{1+\alpha_c^2}\alpha_c\right) a - ig_B A b^* \tag{S73}$$

Thus the α factor for the first Stokes SBL can be defined as,

$$\alpha \equiv \frac{2\delta\Omega}{\Gamma} = \left(\delta\omega - \frac{gN_{a,c}}{1+\alpha_c^2}\alpha_c\right)\left(\frac{\gamma}{2} + \frac{gN_{a,c}}{1+\alpha_c^2}\right)^{-1} \tag{S74}$$

Since the α factors are defined using laser detunings, these are not directly observable in the experiment. Additional phase-matching equations are required to determine the detunings,

$$\delta\Omega + \delta\omega = \frac{\Gamma+\gamma}{2}\alpha' \tag{S75}$$

$$\delta\Omega_c + \delta\omega_c = \frac{\Gamma+\gamma}{2}\alpha'_c + \delta\omega \tag{S76}$$

where $\alpha' \equiv 2(\mathrm{FSR} - \Omega)/(\Gamma+\gamma)$ and $\alpha'_c \equiv 2(\mathrm{FSR}_c - \Omega_c)/(\Gamma+\gamma)$ are the non-cascading α factors, FSR (FSR_c) is the mode spacing between pump and Stokes (Stokes and cascading Stokes), and Ω (Ω_c) is the phonon frequency for the Stokes (cascading Stokes) process. In principle, the above equations, together with Eq. (S74) and Eq. (S70), form a closed set of nonlinear equations that can be solved iteratively. In the limit of $\Gamma \gg \gamma$, the mode pulling effects are weak enough, and we can expand α and α_c to first order of γ/Γ. The results are

$$\alpha = \alpha' + \frac{2}{\Gamma}\frac{gN_{a,c}}{1+(\alpha'_c)^2}(\alpha'_c - \alpha') \tag{S77}$$

$$\alpha_c = \alpha'_c + \frac{\gamma}{\Gamma}\alpha' - \frac{2}{\Gamma}\frac{gN_{a,c}}{1+(\alpha'_c)^2}(\alpha'_c - \alpha') \tag{S78}$$

While the amplitude-phase coupling is given by the α factors above, these factors are no longer related to the SBL noise when cascading. The noise on b_c will be coupled to a due to the anti-Stokes process $a^\dagger a_c b_c + a a_c^\dagger b_c^\dagger$. Using the techniques in Section 3, the linewidth of the SBL can be found, up to zeroth order of γ/Γ, as,

$$\Delta\omega_{\mathrm{SBL}} = \Delta\omega_{\mathrm{SBL},1} + \Delta\omega_{\mathrm{SBL},2} \tag{S79}$$

where the contribution from the first phonon mode reads

$$\Delta\omega_{\mathrm{SBL},1} = \frac{gn_{\mathrm{th}}}{\gamma}\frac{2gN_{a,c} + \gamma(1+\alpha_c^2)}{(1+\alpha_c^2)(2+\alpha^2+\alpha_c^2)^2} \times (\alpha^4 - 4\alpha^3\alpha_c + 2\alpha^2(\alpha_c^2+4) + 4\alpha\alpha_c^3 + \alpha_c^4 + 4) \tag{S80}$$

and the contribution from the second phonon mode via the anti-Stokes process reads

$$\begin{aligned}\Delta\omega_{\mathrm{SBL},2} = {} & \frac{n_{\mathrm{th,c}}}{\gamma}\frac{2}{N_{a,c}(1+\alpha_c^2)(2+\alpha^2+\alpha_c^2)^2} \\ & \times \Big[g^2N_{a,c}^2(\alpha^4 - 4\alpha^3\alpha_c + 2\alpha^2(\alpha_c^2+4) + 4\alpha\alpha_c^3 + \alpha_c^4 + 4) \\ & + 2gN_{a,c}\gamma(\alpha_c^2+1) \\ & \times (-\alpha^3\alpha_c + \alpha^2(\alpha_c^2+2) + \alpha(3\alpha_c^2+2)\alpha_c + \alpha_c^4) \\ & + \gamma^2(\alpha+\alpha_c)^2(1+\alpha_c^2)^3\Big]\end{aligned} \tag{S81}$$

where $n_{\mathrm{th,c}}$ is the number of thermal quanta in the cascading phonon mode. We note that the second contribution has a non-physical divergence with $N_{a,c} \to 0$ when $\alpha + \alpha_c \neq 0$. This is because the noise of amplitude-phase coupling from the anti-Stokes process is not white near the threshold of lasing. A special case is $\alpha = \alpha_c = 0$, which leads to

$$\Delta\omega_{\mathrm{SBL}} = (2gN_{a,c} + \gamma)\frac{gn_{\mathrm{th}}}{\gamma} + 2gN_{a,c}\frac{gn_{\mathrm{th,c}}}{\gamma} \tag{S82}$$

This result is consistent with previous analyses on cascading SBLs [4].

REFERENCES

1. J. Li, H. Lee, T. Chen, and K. J. Vahala, "Characterization of a high coherence, Brillouin microcavity laser on silicon," Opt. express **20**, 20170–20180 (2012).
2. C. Henry, "Theory of the linewidth of semiconductor lasers," IEEE J. Quantum Electron. **18**, 259–264 (1982).
3. H. Lee, T. Chen, J. Li, K. Y. Yang, S. Jeon, O. Painter, and K. J. Vahala, "Chemically etched ultrahigh-Q wedge-resonator on a silicon chip," Nat. Photonics **6**, 369 (2012).
4. R. O. Behunin, N. T. Otterstrom, P. T. Rakich, S. Gundavarapu, and D. J. Blumenthal, "Fundamental noise dynamics in cascaded-order brillouin lasers," Phys. Rev. A **98**, 023832 (2018).